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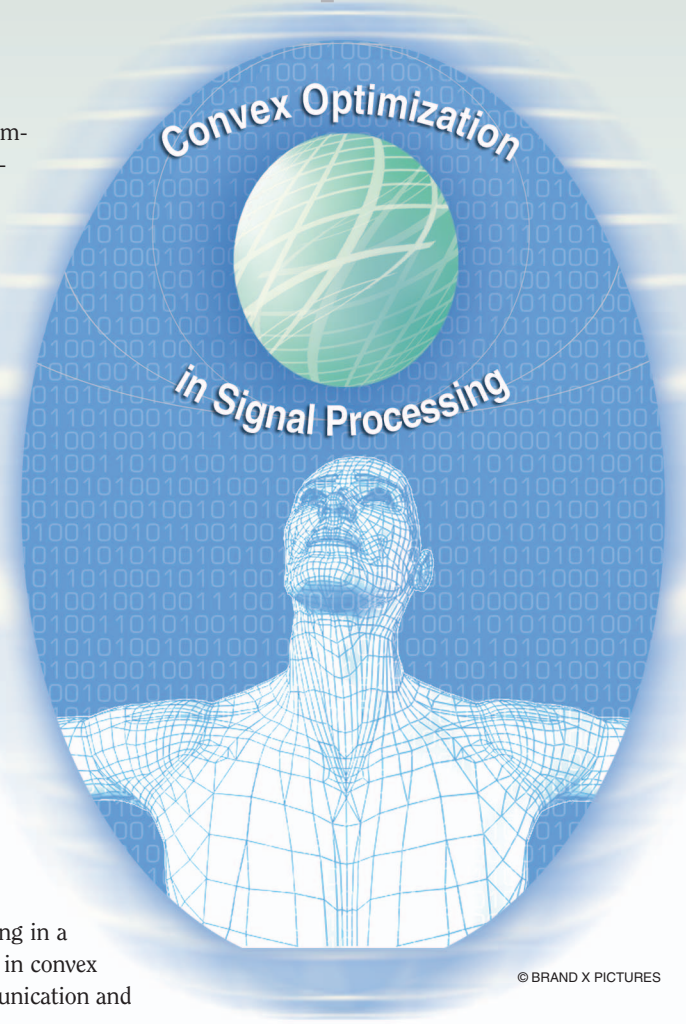
# Convex Optimization, Game Theory, and Variational Inequality Theory

## Basic theoretical foundations and main techniques in multiuser communication systems

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The use of optimization methods is ubiquitous in communications and signal processing. In particular, convex optimization techniques have been widely used in the design and analysis of single user and multiuser communication systems and signal processing algorithms (e.g., [1] and [2]). Game theory is a field of applied mathematics that describes and analyzes scenarios with interactive decisions (e.g., [3] and [4]). Roughly speaking, a game can be represented as a set of coupled optimization problems. In recent years, there has been a growing interest in adopting cooperative and noncooperative game theoretic approaches to model many communications and networking problems, such as power control and resource sharing in wireless/wired and peer-to-peer networks (e.g., [5]–[12]), cognitive radio systems (e.g., [13]–[17]), and distributed routing, flow, and congestion control in communication networks (e.g., [18] and [19] and references therein). Two recent special issues on the subject are [20] and [21]. A more general framework suitable for investigating and solving various optimization problems and equilibrium models, even when classical game theory may fail, is known to be the variation inequality (VI) problem that constitutes a very general class of problems in nonlinear analysis [22].

### MOTIVATION

The goal of this article is twofold. The first half aims at presenting in a unified fashion the theoretical foundations and main techniques in convex optimization, game theory, and VI theory, suitable for the communication and

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signal processing communities. Special emphasis is placed on the generality of the VI framework, showing how several interesting problems in nonlinear analysis, optimization, and equilibrium programming can be formulated as a VI problem, such as nonlinear (convex) optimization problems [22] and (generalized) Nash equilibrium problems [23]. The goal of this first part is to provide the signal processing and communication communities with mathematical tools useful to analyze the basic issue of an equilibrium problem (e.g., existence and uniqueness of the solution) and to devise iterative (possibly) distributed algorithms along with their convergence properties. The second half of the article illustrates how to apply the theoretical results developed in the first part to several equilibrium problems modeling some challenging resource allocation problems in wireless ad hoc or per-to-peer wired networks [6], [8]–[10], in the emerging field of cognitive radio (CR) networks [14], [17], and distributed flow and congestion control problems in multihop communication networks [18], [19]. These applied contexts provide solid evidence of the wide applicability of the VI methodology in modeling and studying further equilibrium problems modeling conflict situations of selfish systems that are relevant to signal processing and communication applications. We hope this article will stimulate the interest in VI theory and its application in the signal processing and communication communities.

## VARIATIONAL INEQUALITIES AND GAME THEORY: BASIC DEFINITIONS AND CONCEPTS

In this section, we provide a short introduction to basic concepts and results about VIs aiming at showing their relevance in the study of games. We also recall concepts of game theory with an emphasis on those that are more relevant to signal processing and communication applications. The machinery discussed in this section will be instrumental to study the resource allocation problems and equilibrium models introduced in the second half of the article.

### *PRELIMINARY BACKGROUND ON CONVEXITY*

We begin recalling a few fundamental definitions about convexity.

#### CONVEX SETS

A set $\mathcal{K} \subseteq \mathbb{R}^n$ is convex if for any two points $\mathbf{x},\ \mathbf{y} \in \mathcal{K}$, the segment joining them belongs to $\mathcal{K}$

$$\alpha\mathbf{x} + (1-\alpha)\mathbf{y} \in \mathcal{K}, \quad \forall \mathbf{x},\ \mathbf{y} \in \mathcal{K} \text{ and } \alpha \in [0,1]. \tag{1}$$

Examples of convex sets include the unit ball $\mathcal{K} = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| \leq 1\}$ (but not the unit sphere $\mathcal{K} = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| = 1\}$), ellipsoids, hypercubes, and polyhedral sets. We recall that the intersection of convex sets is a convex set (while the union of convex sets is not convex, in general). In the real line $\mathbb{R}$, for example, convex set are intervals.

#### CONVEX FUNCTIONS

Given a convex set $\mathcal{K} \subseteq \mathbb{R}^n$ and a function $f(\mathbf{x}): \mathcal{K} \to \mathbb{R}$; $f$ is said to be

- *convex* on $\mathcal{K}$ if, $\forall \mathbf{x}, \mathbf{y} \in \mathcal{K}$ and $\alpha \in (0,1)$,

$$f(\alpha\,\mathbf{x} + (1-\alpha)\,\mathbf{y}) \leq \alpha\, f(\mathbf{x}) + (1-\alpha)\, f(\mathbf{y}) \tag{2}$$

- *strictly convex* on $\mathcal{K}$ if the inequality in (2) is strict
- *strongly convex* on $\mathcal{K}$ if $\forall \mathbf{x}, \mathbf{y} \in \mathcal{K}$ and $\alpha \in (0,1)$, there exists a constant $c > 0$ such that

$$f(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}) \leq \alpha f(\mathbf{x}) + (1-\alpha) f(\mathbf{y}) - \frac{c}{2}\alpha(1-\alpha)\|\mathbf{x}-\mathbf{y}\|^2. \tag{3}$$

Obviously the following relations hold:

strongly convex $\Rightarrow$ strictly convex $\Rightarrow$ convex

but none of the above implications can be reversed in general. The geometric meaning of the definitions above is simple. Consider the function in Figure 1(a) and the segment $\mathcal{S}$ joining the points $(\mathbf{x}, f(\mathbf{x}))$ and $(\mathbf{y}, f(\mathbf{y}))$. Saying that $f$ is convex means that the graph of $f$ lies not above the segment $S$. Strict convexity means that the graph of $f$ lies below the segment $S$, see Figure 1(b); while strong convexity requires the function $f$ to lie "sufficiently" below the segment $S$, see Figure 1(c). A linear function $f(\mathbf{x}) = \mathbf{c}^T\mathbf{x} + \mathbf{b}$ is an example of convex function that is not strictly convex; the exponential $f(x) = e^x$ is a strictly convex function that is not strongly convex; the quadratic function $f(x) = x^2$ is an example of strongly convex function. Many operations on functions preserve convexity, for example, the sum of convex functions, the multiplication of a convex functions by a nonnegative scalar, and the point-wise maximum of convex functions all give rise to convex functions. Many other composition rules that preserves the convexity can be found, e.g., in [24, Ch. 1] and [25, Ch. 3.2].

#### CONVEX OPTIMIZATION PROBLEMS

Consider a generic optimization problem (in the minimization form)

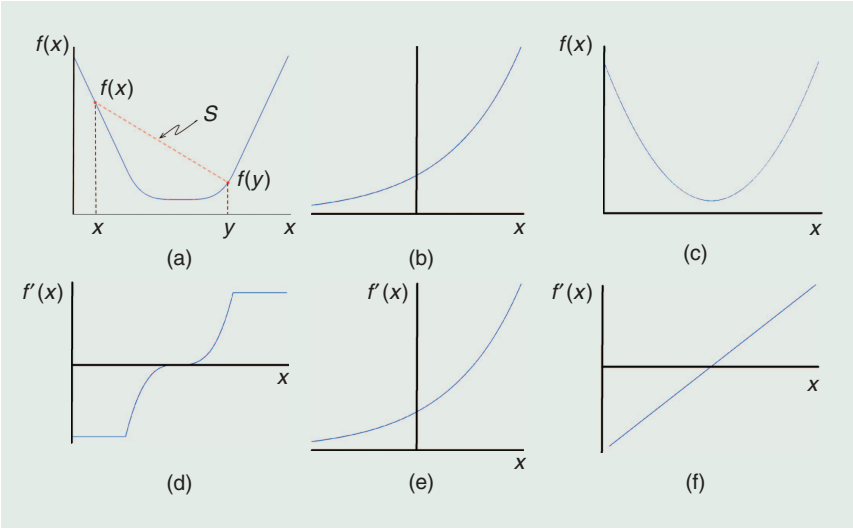
$$\begin{array}{ll} \underset{\mathbf{x}}{\text{minimize}} & f(\mathbf{x}) \\ \text{subject to} & \mathbf{x} \in \mathcal{K}, \end{array} \tag{4}$$

where $f$ is called the objective function (or cost function) and $\mathcal{K}$ is the constraint set. A (feasible) point $\mathbf{x}^\star \in \mathcal{K}$ is said to be optimal if $f(\mathbf{x}^\star) \leq f(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{K}$. We assume throughout that $\mathcal{K}$ is closed and convex and $f$ is convex and continuously differentiable on $\mathcal{K}$; with this assumption the optimization problem above is termed a convex optimization problem.

Convex optimization problems are an important subclass of optimization problems. Their importance stems from the fact that, on the one hand they arise quite frequently in applications and, on the other hand, powerful analytical and

algorithmic tools are available for their study. We refer, e.g., to [24] and [25] for details, but it is safe to say that, to a large extent, convex optimization problems constitute the largest class of tractable optimization problems.

There is a host of important issues that should be addressed in connection to convex optimization problems (e.g., existence of a solution, uniqueness, etc.). We will revisit some of these topics in the next subsection, as a particular case of the study of VIs. Here we only discuss one of the characterizations of optimal solutions that, besides being fundamental in its own right, will be useful to understand the connection between convex optimization problems and VIs to be discussed in the next subsection.




**[FIG1] Some examples of convex and monotone functions: (a) convex function; (b) strictly convex function; (c) strongly convex function; (d) monotone function [first derivative of the convex function in (a)]; (e) strictly monotone function [first derivative of the strictly convex function in (b)]; (f) strongly monotone function [first derivative of the strongly convex function in (c)].**

### Optimality Conditions

Assume that we have a feasible point $\mathbf{x}^\star$: our aim is to understand whether this is an optimal solution, not using the definition, that is hard to verify in practice, but some other conditions that may give some useful insight on the problem and can lead to more tractable conditions. These kind of conditions are called optimality conditions and constitute the foundations for the theoretical study of the problem and its numerical solution. The fundamental optimality conditions for convex optimization problems is called the minimum principle, and we proceed now to its illustration. To understand it properly, recall that the gradient of a (continuously differentiable) function $f$ represents the direction of maximal ascent of the function. By using the Taylor expansion of $f$ around a point $\mathbf{x}$, it is easy to see that if we move slightly from $\mathbf{x}$ along a direction $\mathbf{d}$, then the function values increase with respect to $f(\mathbf{x})$ if $\nabla f(\mathbf{x})^T\mathbf{d} > 0$ (i.e., if $\nabla f(\mathbf{x})$ and $\mathbf{d}$ form an acute angle), decreases if $\nabla f(\mathbf{x})^T\mathbf{d} < 0$ (i.e., if $\nabla f(\mathbf{x})$ and $\mathbf{d}$ form an obtuse angle), while the function behavior cannot be determined using the gradient only, if $\nabla f(\mathbf{x})^T\mathbf{d} = 0$ (i.e., if $\nabla f(\mathbf{x})$ and $\mathbf{d}$ are perpendicular). Therefore the gradient of $f$ at a point $\mathbf{x}$ divides the space into three regions, one in which the function (at least for points close enough to $\mathbf{x}$) increases, one in which the function decreases, and one in which we cannot make a sound guess by using only the gradient; see Figure 2(a). The minimum principle essentially just states that if we consider the convex optimization problem (4) and a feasible point $\mathbf{x}^*$, then, if $\mathbf{x}^*$ is optimal, the feasible region must not lie in the half space where the function decreases; otherwise the point $\mathbf{x}^*$ could not be an optimal solution by definition. It actually turns out that convexity makes this condition also sufficient for optimality. The minimum principle is formally given in (5), while it is illustrated pictorially in Figure 2(b) and (c).

**MINIMUM PRINCIPLE**

Consider the convex optimization problem (4). A feasible point $x^\star \in \mathcal{K}$ is an optimal solution if and only if

$$(\mathbf{y} - \mathbf{x}^\star)^T\nabla f(\mathbf{x}^\star) \geq 0 \quad \forall \mathbf{y} \in \mathcal{K}. \tag{5}$$

Note that if $\mathcal{K} = \mathbb{R}^n$, (5) reduces to the basic necessary (and sufficient for convex $f$) condition for unconstrained optimality of $\mathbf{x}^\star$: $\nabla f(\mathbf{x}^\star) = \mathbf{0}$.

The case in which the set $\mathcal{K}$ is defined by inequalities and equalities deserves a particular attention. In this case it can be shown that, under some additional conditions, the minimum principle is in fact equivalent to the famous Karush-Kuhn-Tucker (KKT) optimality conditions; we refer, e.g., to [24] and [25] for details.

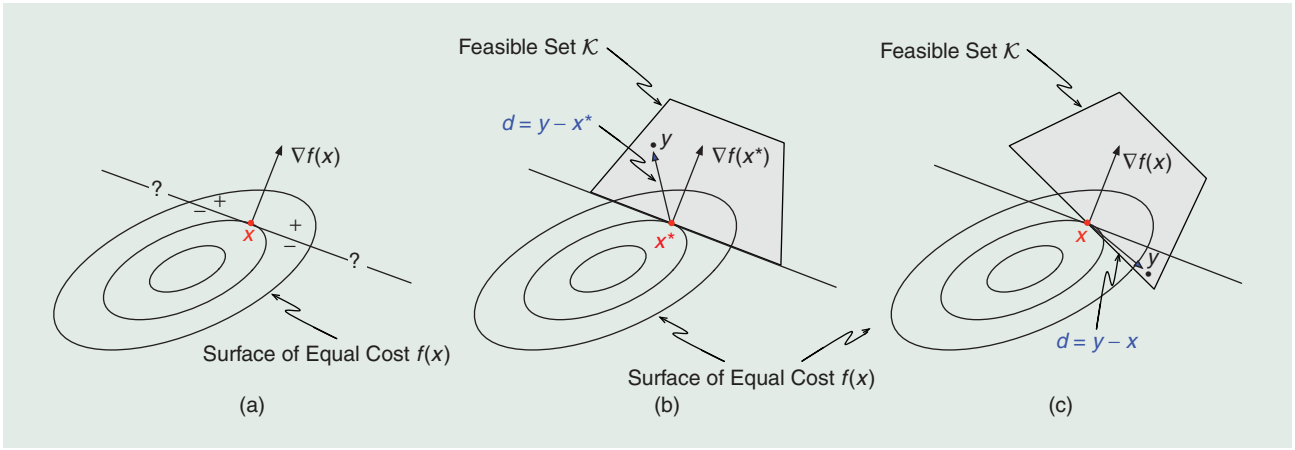
## *VARIATIONAL INEQUALITIES PROBLEMS*

VIs constitute a broad class of problems encompassing convex optimization and bearing strong connections to game theory. The simplest way to see a VI is as a generalization of the minimum principle (5) where the gradient $\nabla f$ is substituted by a general function $\mathbf{F}$. More formally, we have the following.

**VARIATIONAL INEQUALITY PROBLEM**

Given a closed and convex set $\mathbf{F}: \mathcal{K} \subseteq \mathbb{R}^n$ and a mapping $\mathbf{F}: \mathcal{K} \to \mathbb{R}^n$, the VI problem, denoted $\mathrm{VI}(\mathcal{K}, \mathbf{F})$, consists in finding a vector $\mathbf{x}^\star \in \mathcal{K}$ (called a solution of the VI) such that [22]:

$$(\mathbf{y} - \mathbf{x}^\star)^T\mathbf{F}(\mathbf{x}^\star) \geq 0, \quad \forall \mathbf{y} \in K. \tag{6}$$

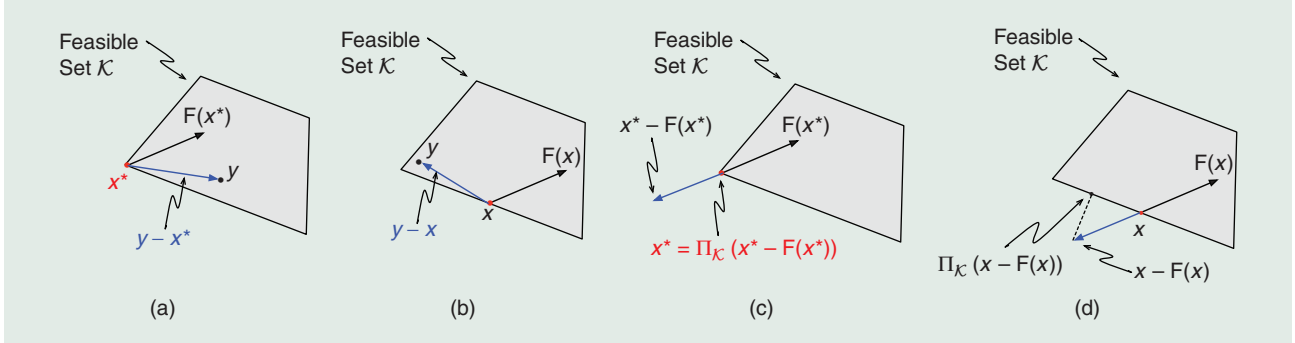



**[FIG2] Geometrical interpretation of the minimum principle: (a) Surfaces of equal cost $x$ with the gradient at $x$ (orthogonal to one of these surfaces) that divides the space into three regions, one in which $f(x)$ (locally) increases (denoted by "+"), one in which $f(x)$ (locally) decreases (denoted by "−"), and one in which we cannot make a sound guess (denoted by "?"). (b) A feasible point $\mathbf{x}^\star$ that satisfies the minimum principle, $\nabla f(x^\star)$ forms a nonobtuse angle with all feasible vectors $d$ emanating from $\mathbf{x}^\star$. (c) A feasible point $x$ that does not satisfy the minimum principle, there are indeed other feasible points $y \neq x$ such that $f(y) < f(x)$.**

In the sequel, for the sake of simplicity, we shall always assume that $\mathbf{F}$ is continuously differentiable on the interior of $\mathcal{K}$ and $\mathcal{K}$ is closed and convex. The geometrical interpretation of (6) is illustrated in Figure 3. It is clear that if $\mathbf{F} = \nabla f$ for some suitable convex function $f$, $\mathrm{VI}(\mathcal{K}, \nabla f)$ coincides with the problem of finding a point satisfying the minimum principle (5) and therefore with the problem of finding an optimal solution of the convex optimization problem (4). However, when $\mathbf{F}$ cannot be expressed as the gradient of some "potential function," the VI is distinct from an optimization problem. It is therefore apparent that VI encompasses a wider range of problems than optimization problems. In fact, we recall that not all continuous functions $\mathbf{F}$ can be expressed as the gradient of a suitable scalar function $f$. It is well known that this happens if and only if the Jacobian matrix of $\mathbf{F}$ is symmetric for all points in the domain of interest. For example, suppose that $\mathbf{F} = \mathbf{A}\mathbf{x} + \mathbf{b}$ for some suitable square $n \times n$ matrix $\mathbf{A}$ and $n$-vector $\mathbf{b}$. If $\mathbf{A}$ is symmetric, it is easy to check that $\mathbf{F}(\mathrm{x}) = \nabla f(\mathrm{x})$, with $f(\mathrm{x}) = (1/2)(\mathrm{x}^T\mathbf{A}\mathrm{x} + \mathrm{b}^T\mathrm{x})$. However, if $\mathbf{A}$ is not symmetric it is impossible to find a function $f$ whose gradient yields $\mathbf{F}$.

The distinction between a convex optimization problem and a VI then essentially boils down to the difference between a VI with an $\mathbf{F}$ that has a symmetric Jacobian or not. At first glance it might seem that there is little gain in relaxing the symmetry condition on the Jacobian of $\mathbf{F}$: this is not so. By allowing functions $\mathbf{F}$ in the definition of VI with a nonsymmetric Jacobian we do get a whole world of new problems and this motivates a detailed study of VIs; we refer to [22] for a detailed discussion on this topic. In the next subsection, we will discuss at length how this provides us with the mathematical background to deal with games. Here we illustrated briefly some other classical problems that fall into the VI framework.

- $\mathcal{K} = \mathbb{R}^n$: *System of equations*. If $\mathcal{K} = \mathbb{R}^n$, then $\mathrm{VI}(\mathbb{R}^n, \mathbf{F})$ is equivalent to finding a $\mathbf{x}^\star \in \mathbb{R}^n$ such that $\mathbf{F}(\mathbf{x}^\star) = \mathbf{0}$, since the only vector $\mathbf{F}(\mathbf{x}^\star)$ which forms a nonobtuse angle with all vectors in $\mathbb{R}^n$ is the zero vector.
- $\mathcal{K} = \mathbb{R}^n_+$: *Nonlinear complementarity problem (NCP)*. When the set $\mathcal{K}$ is the nonnegative orthant of $\mathbb{R}^n$, the VI admits an equivalent form known as a nonlinear complementarity problem, denoted by $\mathrm{NCP}(\mathbf{F})$, which is to find a vector $\mathbf{x}^\star$ such that




**[FIG3] Geometrical interpretation of some basic concepts of VIs: (a) A feasible point $x^\star$ that is a solution of the $VI(\mathcal{K}, F)$, $F(x^\star)$ forms an acute angle with all the feasible vectors $y - x^\star$. (b) A feasible point $x$ that is not a solution of the $VI(\mathcal{K}, F)$. (c) $x^\star$ is a solution of the $VI(\mathcal{K}, F)$ if and only if $x^\star = \prod_{\mathcal{K}}(x^\star - F(x^\star))$ [see (15)]. (d) A feasible $x$ that is not a solution of the $VI(\mathcal{K}, F)$ and thus $x \neq \prod_{\mathcal{K}}(x - F(x))$.**

$$0 \leq \mathbf{x}^\star \perp \mathbf{F}(\mathbf{x}^\star) \geq 0, \tag{7}$$

where $\perp$ means "orthogonal" ($\mathbf{a} \perp \mathbf{b} \Leftrightarrow \mathbf{a}^T\mathbf{b} = 0$). Note that, since $\mathbf{x}^\star \geq \mathbf{0}$ and $\mathbf{F}(\mathbf{x}^\star) \geq \mathbf{0}$, the orthogonality condition in (7) is equivalent to $x_i^\star F_i(\mathbf{x}^\star) = 0$, $\forall i = 1, \ldots n$. The NCP was first identified in the 1964 Ph.D. thesis of R.W. Cottle published in [26] as a unifying mathematical framework for linear programming, quadratic programming, and bimatrix games.

We now focus on the basic issues of existence/uniqueness of a solution and its characterization.

## EXISTENCE AND UNIQUENESS OF THE SOLUTION

The most basic results on the existence of a solution of the VI($\mathcal{K}$, $\mathbf{F}$) is what can be considered as the natural extension of Weierstrass theorem for optimization problems.

Given the VI($\mathcal{K}$, $\mathbf{F}$), suppose that
i) the set $\mathcal{K}$ is convex and compact (closed and bounded);
ii) the function $\mathbf{F}(\mathbf{x})$ is continuous.
Then, the set of solutions is nonempty and compact. (8)

The boundedness assumption of the set $\mathcal{K}$ might be too restrictive (e.g., in the NCP the set is unbounded). Existence can still be established if we trade the boundedness assumption of the set $\mathcal{K}$ with certain additional properties of the function $\mathbf{F}$. To this end we recall some basic "monotonicity" properties of vector functions that are naturally satisfied by the gradient maps of convex functions. Indeed, monotonicity plays in the VI field a role similar to that of convexity in optimization. Given a convex set $\mathcal{K}$, a mapping $\mathbf{F} : \mathcal{K} \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is said to be

- *monotone* on $\mathcal{K}$ if

$$(\mathbf{F}(\mathbf{x}) - \mathbf{F}(\mathbf{y}))^T(\mathbf{x} - \mathbf{y}) \geq 0, \quad \forall \mathbf{x}, \mathbf{y} \in \mathcal{K} \tag{9}$$

- *strictly monotone* on $\mathcal{K}$ if

$$(\mathbf{F}(\mathbf{x}) - \mathbf{F}(\mathbf{y}))^T(\mathbf{x} - \mathbf{y}) > 0, \quad \forall \mathbf{x}, \mathbf{y} \in \mathcal{K} \text{ and } \mathbf{x} \neq \mathbf{y} \tag{10}$$

- *strongly monotone* on $\mathcal{K}$ if there exists a constant $c > 0$ such that

$$(\mathbf{F}(\mathbf{x}) - \mathbf{F}(\mathbf{y}))^T(\mathbf{x} - \mathbf{y}) \geq c\, ||\mathbf{x} - \mathbf{y}||^2, \quad \forall \mathbf{x}, \mathbf{y} \in \mathcal{K}. \tag{11}$$

Figure 1(d)–(f) shows examples of monotone, strictly monotone, and strongly monotone scalar functions. The relations among the above monotonicity properties are the following: strongly monotone $\Rightarrow$ strictly monotone $\Rightarrow$ monotone. There also exists a connection between the above monotonicity properties and the positive semidefiniteness of the Jacobian matrix of $\mathbf{F}$; we refer to [22, Ch. 2] for the details. For the important class of affine functions, $\mathbf{F}(\mathbf{x}) = \mathbf{A}\,\mathbf{x} + \mathbf{b}$, where $\mathbf{A}$ is an $n \times n$ (not necessarily symmetric) matrix and $\mathbf{b}$ is an $n$-vector, some stronger results are valid. $\mathbf{F}(\mathbf{x}) = \mathbf{A}\,\mathbf{x} + \mathbf{b}$ is monotone if and only if $\mathbf{A}$ is positive semidefinite, whereas the strict and strong monotonicity are equivalent among themselves and to the positive definiteness of $\mathbf{A}$. Finally, observe that if the vector function $\mathbf{F}$ is the gradient of a scalar function $f$ (denoted by $\nabla f$), the above monotonicity properties can be related to the convexity properties of the function $f$ discussed in the previous subsection.

| | | |
|---|---|---|
| i) $f$ convex | $\Leftrightarrow$ | $\nabla f$ monotone |
| ii) $f$ strictly convex | $\Leftrightarrow$ | $\nabla f$ strictly monotone |
| iii) $f$ strongly convex | $\Leftrightarrow$ | $\nabla f$ strongly monotone |

(12)

Figure 1 shows an example of the relationship above between the convexity properties of a scalar function and the monotonicity properties of its derivative. Using the above monotonicity properties, we can now state a few results on the solutions of the VI($\mathcal{K}$, $\mathbf{F}$) without requiring the boundedness of the (closed and convex) set $\mathcal{K}$ (recall that $\mathbf{F}$ is assumed to be continuous on $\mathcal{K}$).

i) If $\mathbf{F}$ is monotone on $\mathcal{K}$, the solution set of the VI($\mathcal{K}$, $\mathbf{F}$) is closed and convex.
ii) If $\mathbf{F}$ is strictly monotone on $\mathcal{K}$, the VI($\mathcal{K}$, $\mathbf{F}$) admits at most one solution.
iii) If $\mathbf{F}$ is strongly monotone on $\mathcal{K}$, the VI($\mathcal{K}$, $\mathbf{F}$) admits a unique solution. (13)

Note that the strict monotonicity of $\mathbf{F}$ on $\mathcal{K}$ does not guarantee the existence of a solution of the VI($\mathcal{K}$, $\mathbf{F}$). For example, $F(x) = e^x$ is a strictly monotone function but the VI($\mathbb{R}$, $e^x$) does not have solutions. The results above allow us to recover standard results on the existence and uniqueness of a solution of convex optimization problems. For example, it follows from iii) of (13) that the VI($\mathcal{K}$, $\nabla f$) admits a unique solution if $\nabla f$ is strongly monotone which, using iii) of (12), is equivalent to state that the convex optimization problem (4) admits a unique solution if $f$ is strongly convex. It is also possible to give several further conditions for the existence of solutions of VIs with unbounded feasible sets; we refer the reader to [22, Sec. 2].

## CHARACTERIZATION OF THE SOLUTION

Several equivalent formulations of the VI problem and thus characterizations of the solution can be found in the literature in terms of systems of equations and/or optimization problems of various kinds [22, Sec. 1.5]. Such formulations can be very useful for both analytical and computational purposes. Here we focus on the reformulation of the VI problem as a classical fixed-point problem, which paves the way for the development of a large family of iterative methods, some of them used in the second part of the article. The fixed-point based reformulation involves the Euclidean projection onto a closed convex set, which is defined next. The Euclidean projection of a vector $\mathbf{x}_0$ onto a closed and convex set $\mathcal{K}$, denoted $\prod_{\mathcal{K}}(\mathbf{x}_0)$, is the unique vector in $\mathcal{K}$ that is closest to $\mathbf{x}_0$ in the Euclidean norm. By definition, $\prod_{\mathcal{K}}(\mathbf{x}_0)$ is the unique solution of the following convex minimization problem (note that the

objective function is strongly convex and thus the solution exists and is unique), where $\mathbf{x}_0$ is considered fixed

$$\begin{array}{ll} \underset{\mathbf{y}}{\text{minimize}} & ||\mathbf{y}-\mathbf{x}_0||^2 \\ \text{subject to} & \mathbf{y}\in\mathcal{K}. \end{array} \tag{14}$$

The connection with the $\text{VI}(\mathcal{K}, \mathbf{F})$ is the following:

$$\mathbf{x}^\star \text{ is a solution of the VI}(\mathcal{K},\mathbf{F}) \;\Leftrightarrow\; \mathbf{x}^\star=\prod_{\mathcal{K}}(\mathbf{x}^\star-\mathbf{F}(\mathbf{x}^\star)). \tag{15}$$

The equivalence in (15) can be easily understood geometrically, as shown in Figure 3(c) and (d).

As for the classical convex optimization problems, there are KKT conditions also for the $\text{VI}(\mathcal{K}, \mathbf{F})$; we refer to [22] for details.

Several solution methods for VIs have been proposed in the literature. A treatment on the subject goes beyond the scope of this article and we refer the interest reader to the technical literature on the subject. A good entry point on parallel and distributed algorithms and their convergence for optimization problems and variational inequalities is the book [27]. A comprehensive and more advanced treatment can be found in the monograph [22]. In the second part of the article, we specialize some of these algorithms to solve the proposed equilibrium problems in multiuser communication systems.

## *NONCOOPERATIVE GAMES*

Noncooperative game theory is a branch of game theory for the resolution of conflicts among interacting decision makers (called players), each behaving selfishly to optimize one's own well being, quantified in general through an objective function. While many problems in signal processing and communications have traditionally been approached by using optimization, game models are being increasingly used. They seem to provide meaningful models for many applications where the interaction among several agents is by no means negligible and centralized approaches are not suitable, e.g., in emerging wireless networks, such as sensor networks, ad hoc networks, CR systems, and pervasive computing systems. Furthermore, the deregulation of telecommunication markets and the explosive growth of the Internet pose many new problems that can be effectively tackled with game-theoretic tools.

In this section, we consider two classes of problems. The first is the class of Nash equilibrium problems (NEPs) where the interactions among players take place at the level of objective functions only. The second is the class of generalized NEPs (GNEPs) where in addition we have that the choices available to each player also depend by the actions taken by his rivals. The NEP is by far better studied and "easier." The GNEP has a wider range of applicability but sparser results are available for its study.

### NASH EQUILIBRIUM PROBLEMS

Assume there are $Q$ players each controlling the variables $\mathbf{x}_i \in \mathbb{R}^{n_i}$. We denote by $\mathbf{x}$ the overall vector of all variables: $\mathbf{x} \triangleq (\mathbf{x}_1, \ldots, \mathbf{x}_Q)$; while we use the notation $\mathbf{x}_{-i} \triangleq (\mathbf{x}_1, \ldots, \mathbf{x}_{i-1}, \mathbf{x}_{i+1}, \ldots, \mathbf{x}_Q)$ to denote the vector of all players' variables except that of player $i$. The aim of player $i$, given the other players' strategies $\mathbf{x}_{-i}$, is to choose an $\mathbf{x}_i \in \mathcal{Q}_i$ that minimizes his payoff function $f_i(\mathbf{x}_i, \mathbf{x}_{-i})$, i.e.,

$$\begin{array}{ll} \underset{\mathbf{x}_i}{\text{minimize}} & f_i(\mathbf{x}_i, \mathbf{x}_{-i}) \\ \text{subject to} & \mathbf{x}_i \in \mathcal{Q}_i. \end{array} \tag{16}$$

Roughly speaking, an NEP is a set of coupled optimization problems. We make the blanket assumption that the objective functions $f_i$ are continuously differentiable and, as a function of $\mathbf{x}_i$ alone, convex, while the sets $\mathcal{Q}_i \subseteq R^{n_i}$ are all closed and convex. A point $\mathbf{x}$ is feasible if $\mathbf{x}_i \in \mathcal{Q}_i$ for all players $i$. A (pure strategy) NE, or simply a solution of the NEP, is a feasible point $\mathbf{x}^\star$ such that

$$f_i(\mathbf{x}_i^\star, \mathbf{x}_{-i}^\star) \le f_i(\mathbf{x}_i, \mathbf{x}_{-i}^\star), \quad \forall \mathbf{x}_i \in \mathcal{Q}_i \tag{17}$$

holds for each player $i = 1, \ldots, Q$. In words, an NE is a feasible strategy profile $\mathbf{x}^\star$ with the property that no single player can benefit from a unilateral deviation from $\mathbf{x}_i^\star$, if all the other players act according to it.

In general, the existence of an NE as defined in (17) is not guaranteed; neither are the uniqueness nor the convergence (e.g., of best-response-based algorithms) to an equilibrium when one exists (or even is unique). To address these key issues, a useful way to see an NE is as a fixed point of the best-response mapping for each player. Let $\mathcal{B}_i(\mathbf{x}_{-i})$ be the set of optimal solutions of the $i$th optimization problem (16) and set $\mathcal{B}(\mathbf{x}) \triangleq B_1(\mathbf{x}_{-1}) \times B_2(\mathbf{x}_{-2}) \times \cdots \times B_Q(\mathbf{x}_{-Q})$. It is clear that a point $\mathbf{x}^\star$ is an NE if and only if it is a fixed point of $\mathcal{B}(\mathbf{x})$, i.e., if and only if $\mathbf{x}^\star \in \mathcal{B}(\mathbf{x}^\star)$. This observation is the key to the standard approach to the study of NEPs: the so-called fixed-point approach, which is based on the use of the well-developed machinery of fixed-point theory. This approach is adopted in the analysis of several games proposed in the signal processing and communication literature to model challenging resource allocation problems in wireless single-input, single-output (SISO)/multiple-input, multiple-output (MIMO) ad hoc or peer-to-peer wired networks [5], [7]–[12] and in the emerging field of CR networks [14]–[16]. Some of these games will be analyzed in the section "Nash Equilibrium Problems: Rate Maximization Game Over Parallel Gaussian Interference Channels." However, the applicability of the fixed-point based analysis as used in the aforementioned papers requires the ability to compute the best-response mapping $\mathcal{B}(\mathbf{x})$ in closed form; this feature may not be an easy task for a game with arbitrary payoff functions and strategy sets, which certainly strongly limits the applicability of this methodology.

There are at least two other ways to study NEPs. The first is based on a reduction of the NEP to a VI. This approach is pursued in detail in [28] and, resting on the well-developed theory of VIs, has the advantage of permitting an easy derivation of many results about existence, uniqueness, and stability of the solutions. But its main benefit is probably that of leading quite naturally to the derivation of implementable solution algorithms along with their convergence properties. It is this approach that will be at the basis of our

exposition and will be exemplified in the next subsection. The second alternative approach is based on an ad hoc study of classes of games having a particular structure that can be exploited to facilitate their analysis. For example, this is the case of the so-called potential games [29] and supermodular games [30]. These classes of games have recently received great attention in the signal processing and communication communities as a useful tool to model and solve various power control problems in wireless communications and networking [18], [31], [32]. As an example, in the second part of the article, we show how a fairly general class of distributed flow and congestion control problems fit naturally in the framework of potential games and, building on the structure of the game, we propose a distributed algorithm that converges to an NE.

### VI Reformulation of the NEP

At the basis of the VI approach to NEPs there is an easy equivalence between an NEP and a suitably defined VI. In fact, given the equivalence between the VI problem and a convex optimization problem (cf. the section "Variational Inequalities Problems"), the following result follows readily from the minimum principle (5) for convex problems. In what follows, we denote by $\mathcal{G} = \langle Q, \mathbf{f} \rangle$ the game defined by the problems (16), with the understanding that $\mathcal{Q} \triangleq \prod_{i=1}^{Q} Q_i$ and $\mathbf{f} \triangleq (f_i(\mathbf{x}))_{i=1}^{Q}$.

Given the game $\mathcal{G} = \langle Q, \mathbf{f} \rangle$, suppose that for each player $i$
- i) the strategy set $\mathcal{Q}_i$ is closed and convex;
- ii) the payoff function $f_i(\mathbf{x}_i, \mathbf{x}_{-i})$ is continuously differentiable in $\mathbf{x}$ and convex in $\mathbf{x}_i$ for every fixed $\mathbf{x}_{-i}$.

Then, the game $\mathcal{G}$ is equivalent to the VI($\mathcal{Q}$, F), where $\mathbf{F}(\mathbf{x}) \triangleq (\nabla_{\mathbf{x}_i} f_i(\mathbf{x}))_{i=1}^{Q}$. (18)

Indeed, each problem (16) is a convex programming problem for each $i$. Therefore, given a feasible $\mathbf{x}^\star$, each $\mathbf{x}_i^\star$ is an optimal solution of (16) if and only if it satisfies the minimum principle [see (5)]: $(\mathbf{y}_i - \mathbf{x}_i^\star)^T \nabla_{\mathbf{x}_i} f_i(\mathbf{x}_i^\star, \mathbf{x}_{-i}^\star) \geq 0$, for all $\mathbf{y}_i \in \mathcal{Q}_i$. Summing these conditions and taking into account the Cartesian product structure of $\mathcal{Q}$, leads to the desired equivalence between the NEP and the VI problem.

### Existence and Uniqueness of the NE Based on VI

Given the equivalence between the NEP and the VI problem, conditions guaranteeing the existence of an NE follow readily from the existence of a solution of the VI: Suppose that, in addition to conditions i) and ii) in (18), each player's strategy set $\mathcal{Q}_i$ is compact, then the NEP has a convex and nonempty solution set, thanks to the existence results (13). Further existence results for unbounded feasible sets can also be obtained by using the VI approach, we refer to [28] for the details. As far as the uniqueness of the NE is concerned, sufficient conditions come from iii) of (13): Assuming that the function $\mathbf{F}(\mathbf{x}) \triangleq (\nabla_{\mathbf{x}_i} f_i(\mathbf{x}))_{i=1}^{Q}$ is strongly monotone on $\mathcal{Q}$, we immediately have that $\mathcal{G} = \langle Q, \mathbf{f} \rangle$ has a unique solution. Sufficient conditions easily to be checked that guarantees such a $\mathbf{F}$ being strongly monotone on $\mathcal{Q}$ are given in [17] and [28].

**ALGORITHM 1: GAUSS-SEIDEL BEST RESPONSE-BASED ALGORITHM**

(S.0): Choose any feasible starting point $\mathbf{x}^{(0)} = (\mathbf{x}_i^{(0)})_{i=1}^{Q}$, and set $n = 0$.

(S.1): If $\mathbf{x}^{(n)}$ satisfies a suitable termination criterion: STOP

(S.2): for $i = 1, \ldots, Q$, compute a solution $\mathbf{x}_i^{(n+1)}$ of

$$\underset{\mathbf{x}_i}{\text{minimize}} \quad f_i(\mathbf{x}_1^{(n+1)}, \ldots, \mathbf{x}_{i-1}^{(n+1)}, \mathbf{x}_i, \mathbf{x}_{i+1}^{(n)}, \ldots, \mathbf{x}_Q^{(n)}) \quad \text{subject to} \quad \mathbf{x}_i \in \mathcal{Q}_i, \tag{19}$$

end

(S.3): Set $x^{(n+1)} \triangleq (x_i^{(n+1)})_{i=1}^{Q}$ and $n \leftarrow n + 1$; go to (S.1).

### Algorithms for Nash Equilibria

Building on the equivalence between the NEP and the VI problem, one can borrow solutions methods for the NEP from the vast literature on variational inequalities (e.g., [22, Ch. 9–12]). For the purposes of this article, we restrict our attention to distributed algorithms. Since in a Nash game every player is trying to minimize his own objective function, a natural approach is to consider an iterative algorithm based, e.g., on the Jacobi (simultaneous) or Gauss-Seidel (sequential) schemes, where at each iteration every player, given the strategies of the others, updates his own strategy by solving his optimization problem (16). The Gauss-Seidel implementation of the best-response-based algorithm is formally described in Algorithm 1. Building on the VI framework, one can prove that Algorithm 1, as well as its Jacobi version, globally converge to the NE of the game, under the same conditions guaranteeing the uniqueness of the equilibrium [the strong monotonicity of $\mathbf{F}$ defined in (18)], as given in [28] and [17].

In many practical multiuser communication systems, such as wireless ad hoc networks or CR systems, the synchronization requirements imposed by the sequential and simultaneous algorithms described above might be not always acceptable. It is possible to show that under mild conditions a totally asynchronous implementation (in the sense of [27]) converges to the unique NE of the game (see, e.g., [9], [16], and [33] for details). Some instances of the above algorithms will be discussed in the second part of the article in the context of decentralized power control problems in wired/wireless multiuser communication systems.

### Nash Equilibria and Pareto Optimality

An alternative, widely used solution concept for problems with multiple decision makers is that of Pareto efficiency. A strategy profile $\mathbf{x} \in \mathcal{Q}$ is Pareto efficient (optimal) if there exists no other strategy $\mathbf{y} \in \mathcal{Q}$ such that $f_i(\mathbf{y}) \leq f_i(\mathbf{x})$ for all $i = 1, \ldots, Q$, and $f_j(\mathbf{y}) < f_j(\mathbf{x})$ for at least one $j$; this is a sort of "social-type optimality." It would obviously be desirable that an NE of the game would also be Pareto efficient. Unfortunately, even when the NE is unique, it need not be Pareto efficient. This obviously raises the question of the suitability of the NE as a conceptual solution in many scenarios where the main objective should be that of

maximizing some sort of collective welfare, as it is often the case for the kind of problems we analyze in this article. The reasons to accept the NE as a desirable outcome is that, in general, Pareto efficiency can only be achieved by performing some kind of centralized (often nonconvex) optimization that is simply physically not plausible in many practical applications in signal processing and communications as, e.g., sensor and ad hoc networks and CR systems. The NE solutions, instead, are better suited for distributed computation without requiring exchange of information among the players. There are also some scenarios where a system-wide optimization cannot be implemented as the players model heterogeneous systems that are not willing to cooperate. A comparison of the performance achievable by noncooperative (decentralized) and cooperative (centralized) solutions in the context of wireless communication networks and CR can be found in [8] and [21].

### GENERALIZED NASH EQUILIBRIUM PROBLEMS

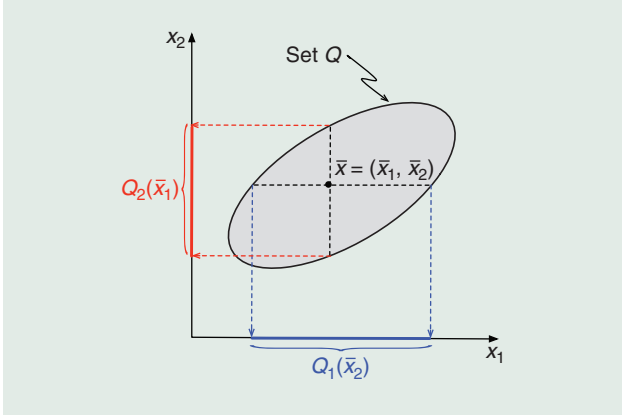
The GNEP extends the classical NEP described so far by assuming that each player's strategy set can depend on the rival players' strategies $\mathbf{x}_{-i}$, so we will write $\mathcal{Q}_i(\mathbf{x}_{-i})$ to indicate that we might have a different closed convex set $\mathcal{Q}_i$ for each different $\mathbf{x}_{-i}$. Analogously to the NEP case, the aim of each player $i$, given $\mathbf{x}_{-i}$, is to choose a strategy $\mathbf{x}_i \in \mathcal{Q}_i(\mathbf{x}_{-i})$ that solves the problem

$$\begin{aligned} \underset{\mathbf{x}_i}{\text{minimize}} \quad & f_i(\mathbf{x}_i, \mathbf{x}_{-i}) \\ \text{subject to} \quad & \mathbf{x}_i \in \mathcal{Q}_i(\mathbf{x}_{-i}). \end{aligned} \tag{20}$$

A generalized NE (GNE) is a tuple of strategies $\mathbf{x}^\star = (\mathbf{x}_1^\star, \ldots, \mathbf{x}_Q^\star)$ such that, for all $i = 1, \ldots, Q$,

$$f_i(\mathbf{x}_i^\star, \mathbf{x}_{-i}^\star) \le f_i(\mathbf{x}_i, \mathbf{x}_{-i}^\star), \quad \forall \mathbf{x}_i \in \mathcal{Q}_i(\mathbf{x}_{-i}^\star). \tag{21}$$

The requirement that the feasible sets depend on the variables of players' rivals is natural in many applications, for example, think of the case in which the players share some common resource, such as a bandwidth, the capacity of a communication link, or a time slot. In the section "Generalized Nash Equilibrium Problems: Power Minimization Game with Quality of Service Constraints Over Interference Channels," we consider a GNEP model representing some power control problems in ad-hoc wireless networks. A survey on GNEPs, with much historical information, is given in [23].




**[FIG4] Example of sets $\mathcal{Q}$ and $\mathcal{Q}_i(\mathbf{x}_{-i})$ for a GNEP with shared constraints defined in (22) and (23), respectively.**

Due to the variability of the feasible sets, the GNEP is a much harder problem than an ordinary NEP. Indeed in its full generality the GNEP problem is almost intractable and also the VI approach is of no great help. But if we restrict our attention to particular classes of problems meaningful results can still be obtained. In the section "Generalized Nash Equilibrium Problems: Power Minimization Game with Quality of Service Constraints Over Interference Channels," we deal with a GNEP with a specific structure that, through a nontrivial transformation can be turned into an NEP and thus studied using the VI framework. In the remaining part of this section we consider the important class of so-called GNEPs with shared constraints, a class of equilibrium problems with many practical applications (see the sections "VI Formulation: Design of CR Systems Under Temperature-Interference Constraints" and "Potential Games: Flow and Congestion Control in Multihop Communication Networks").

#### GNEPs with Shared Constraints

A GNEP is termed a GNEP with shared constraints if the feasible sets $\mathcal{Q}_i(\mathbf{x}_{-i})$ are defined as

$$\mathcal{Q}_i(\mathbf{x}_{-i}) \triangleq \{\mathbf{x}_i \in \mathcal{K}_i : \mathbf{g}(\mathbf{x}_i, \mathbf{x}_{-i}) \le \mathbf{0}\},$$

where $\mathcal{K}_i$ is the (closed and convex) set of individual constraints of player $i$ and $\mathbf{g}(\mathbf{x}_i, \mathbf{x}_{-i}) \le \mathbf{0}$ represents the set of shared coupling constraints (equal for all the players), with $\mathbf{g} = (g_j)_{j=1}^{m_i}$ assumed to be continuously differentiable and (jointly) convex in $\mathbf{x}$. Note that if there are no coupling constraints, the problem reduces to a standard NEP.

We can give a nice geometric interpretation to the conditions above. For a GNEP with shared constraints, let us define

$$\mathcal{Q} \triangleq \{\mathbf{x} : \mathbf{g}(\mathbf{x}_i, \mathbf{x}_{-i}) \le \mathbf{0}, \quad \mathbf{x}_i \in \mathcal{K}_i \ \ \forall i = 1, \ldots, Q\}. \tag{22}$$

It is easy to check that the closed set $\mathcal{Q}$ is convex (thanks to the joint convexity) and that (thanks to the fact that the coupling constraints are the same for all players) we can write

$$\begin{aligned} \mathcal{Q}_i(\mathbf{x}_{-i}) &= \{\mathbf{x}_i \in \mathcal{K}_i : \mathbf{g}(\mathbf{x}_i, \mathbf{x}_{-i}) \le \mathbf{0}\} \\ &= \{\mathbf{x}_i : (\mathbf{x}_i, \mathbf{x}_{-i}) \in \mathcal{Q}\}. \end{aligned} \tag{23}$$

Figure 4 illustrates this construction. GNEPs with shared constraints are still very difficult problems, however at least some types of solutions can be studied and calculated relatively easily by using a VI approach. To this end define as usual the function $\mathbf{F}(\mathbf{x}) \triangleq (\nabla_{\mathbf{x}_i} f_i(\mathbf{x}))_{i=1}^{Q}$ and consider the VI$(\mathcal{Q}, \mathbf{F})$, with $\mathcal{Q}$ defined in (22). It can be seen that every solution of this VI is a solution of GNEP with shared constraints, but not vice versa [34], [4]; the numerical example below illustrates this fact. The solutions of the GNEP that are also solutions of the VI$(\mathcal{Q}, \mathbf{F})$ are termed "variational solutions" or "normalized solutions." Since these variational solutions are solutions of a VI, we can proceed as we did in the previous

subsections and easily derive existence and uniqueness results. It is also possible to develop centralized algorithms, since we can use any method for the solution of the VI$(\mathcal{Q}, \mathbf{F})$. What is more problematic though is the development of distributed algorithms, since in this case the variability of the feasible sets complicates considerably the analysis. We will come back to this in the sections "VI Formulation: Design of CR Systems Under Temperature-Interference Constraints" and "Potential Games: Flow and Congestion Control in Multihop Communication Networks."

**Example of GNEP with Infinite Solutions and One Variational Solution**

Consider the GNEP with two players

$$\begin{array}{ll} \underset{x}{\text{minimize}} & (x-1)^2 \\ \text{subject to} & x+y \le 1 \end{array} \qquad \begin{array}{ll} \underset{y}{\text{minimize}} & (y-\frac{1}{2})^2 \\ \text{subject to} & x+y \le 1. \end{array} \tag{24}$$

It can be shown that this game has infinitely many solutions given by $(\alpha,\ 1-\alpha)$ for every $\alpha \in [1/2,\ 1]$. The VI associated to the GNEP (24) is VI$(\mathcal{Q}, \mathbf{F})$, with $\mathcal{Q} = \{(x, y) \in R^2 : x + y \le 1\}$ and $\mathbf{F} = (2x - 2,\ 2y - 1)^T$, which admits a unique solution given by $(3/4,\ 1/4)$ [note that $\mathbf{F}$ is strongly monotone, see iii) in (13)]. We see then that while the GNEP has infinitely many solutions, the variational solution is unique.

Variational solutions are particularly useful in many applications since they have an interesting "economic" interpretation. Indeed, it can be shown that $\bar{\mathbf{x}}$ is a variational solution if and only if $\bar{\mathbf{x}}$, along with a suitable $\bar{\boldsymbol{\lambda}}$ satisfies the NEP defined by

$$\begin{array}{ll} \underset{\mathbf{x}_i}{\text{minimize}} & f_i(\mathbf{x}_i, \mathbf{x}_{-i}) + \sum_{k=1}^{m} \bar{\lambda}_k g_k(\mathbf{x}_i, \mathbf{x}_{-i}) \\ \text{subject to} & \mathbf{x}_i \in \mathcal{K}_i, \end{array} \tag{25}$$

$\forall i = 1, \ldots, Q,$ and furthermore

$$0 \le \bar{\boldsymbol{\lambda}} \perp \mathbf{g}(\bar{\mathbf{x}}) \le 0. \tag{26}$$

The NEP (25) may be seen as a penalized version of the original GNEP, where we attempt to enforce the shared constraints by making the players pay the price $\bar{\boldsymbol{\lambda}}$ so that $\bar{\boldsymbol{\lambda}}$ can be interpreted as the common prices that players should pay for the resources represented by these constraints. In the section "VI Formulation: Design of CR Systems Under Temperature-Interference Constraints," we show that this pricing mechanism is the natural scheme for modeling concurrent communications among primary and secondary users in a CR system, where the primary users need to control the interference generated by the secondary users in a distributed fashion.

## APPLICATION OF VI TO THE ANALYSIS OF MULTIUSER COMMUNICATION SYSTEMS

In this section, we show how to apply the VI framework developed so far to solve several recent resource allocation equilibrium problems in peer-to-peer [ad hoc and digital subscriber lines (DSLs)] networks, CR systems, and multihop networks.

### *NASH EQUILIBRIUM PROBLEMS: RATE MAXIMIZATION GAME OVER PARALLEL GAUSSIAN INTERFERENCE CHANNELS*

We consider a $Q$-user $N$-parallel Gaussian interference channel (IC). In this model, there are $Q$ transmitter-receiver pairs, where each transmitter wants to communicate with its corresponding receiver over a set of $N$ parallel Gaussian subchannels, that may represent time or frequency bins (here we consider transmissions over a frequency-selective IC, without loss of generality). We denote by $H_{ij}(k)$ the (cross-) channel transfer function over the $k$th frequency bin between the transmitter $j$ and the receiver $i$, while the channel transfer function of link $i$ is $H_{ii}(k)$. The transmission strategy of each user (pair) $i$ is the power allocation vector $\mathbf{p}_i = \{p_i(k)\}_{k=1}^N$ over the $N$ subcarriers, subject to the transmit power constraint

$$\mathcal{P}_i \triangleq \left\{ \mathbf{p} \in \mathbb{R}_+^N : \sum_{k=1}^{N} p(k) \le P_i \right\}. \tag{27}$$

Spectral mask constraints $\mathbf{p}_i^{\max} = (p_i^{\max}(k))_{k=1}^N$ in the form $\mathbf{0} \le \mathbf{p} \le \mathbf{p}^{\max}$ can also be included in the set $\mathcal{P}_i$ (see [6], [8], and [9] for more general results). Under basic information theoretical assumptions (see, e.g., [5] and [8]), the maximum achievable rate on link $i$ for a specific power allocation profile $\mathbf{p}_1, \ldots, \mathbf{p}_Q$ is

$$r_i(\mathbf{p}_i, \mathbf{p}_{-i}) = \sum_{k=1}^{N} \log\left(1 + \frac{|H_{ii}(k)|^2 p_i(k)}{\sigma_i^2(k) + \sum_{j \ne i} |H_{ij}(k)|^2 p_j(k)}\right), \tag{28}$$

where $\mathbf{p}_{-i} \triangleq (\mathbf{p}_1, \ldots, \mathbf{p}_{i-1}, \mathbf{p}_{i+1}, \ldots, \mathbf{p}_Q)$ is the set of all the users power allocation vectors, except the $i$th one, and $\sigma_i^2(k) + \sum_{j \ne i} |H_{ij}(k)|^2 p_j(k)$ is the overall power spectral density (PSD) of the noise plus multiuser interference (MUI) at each subcarrier measured by the receiver $i$.

Given the above setup, we consider the following NEP [6], [8]–[10], [35] (see, e.g., [8] and [21] for a discussion on the relevance of this game theoretical model in practical multiuser systems, such as DSLs, wireless ad hoc networks, peer-to-peer systems, and multicell orthogonal frequency-division multiplexing/time division multiple access cellular systems)

$$\begin{array}{ll} \underset{\mathbf{p}_i}{\text{maximize}} & r_i(\mathbf{p}_i, \mathbf{p}_{-i}) \\ \text{subject to} & \mathbf{p}_i \in \mathcal{P}_i, \end{array} \tag{29}$$

for all $i = 1, \ldots, Q$, where $\mathcal{P}_i$ and $r_i(\mathbf{p}_i, \mathbf{p}_{-i})$ are defined in (27) and (29), respectively. We show next how to study the NEP (29) using the VI framework described in the first part of the article.

#### VI REFORMULATION

The NEP (29) can obviously be rewritten as a VI$(\mathcal{Q}, \mathbf{F})$ as shown in (18) (cf. the section "VI Reformulation of the NEP"), with

$\mathcal{Q} = \mathcal{P}_1 \times \cdots \times \mathcal{P}_Q$ and $\mathbf{F}(\mathbf{p}) \triangleq (-\nabla_{\mathbf{p}_i} r_i(\mathbf{p}_i, \mathbf{p}_{-i}))_{i=1}^{Q}$. Note however that this VI has a nonlinear $\mathbf{F}$. Interestingly, in the case of game (29), it is also possible to give the alternative VI formulation with a linear $\mathbf{F}$, which turns out to be very useful in simplifying the analysis of the game, especially the study of convergence of iterative algorithms. We illustrate this formulation shortly.

First of all, observe that, for any fixed $\mathbf{p}_{-i} \geq \mathbf{0}$, the single-user optimization problem in (29) admits a unique solution (indeed, the feasible set is convex and compact and $r_i(\mathbf{p}_i, \mathbf{p}_{-i})$ is strictly concave in $\mathbf{p}_i \in \mathcal{P}_i$; see the section "Variational Inequalities Problems")], given by the well-known waterfilling expression

$$p_i^\star(k) = [\mathrm{wf}_i(\mathbf{p}_{-q})]_k \triangleq \left[\mu_i - \frac{\sigma_i^2(k) + \sum_{j \neq i} |H_{ij}(k)|^2 p_j(k)}{|H_{ii}(k)|^2}\right]^+, \tag{30}$$

with $k = 1, \ldots, N$, where $[x]^+ \triangleq \max(0, x)$ and the water level $\mu_i$ is chosen to satisfy the transmit power constraint $\sum_{k=1}^{N} p_i^\star(k) = P_i$. The Nash equilibria $\mathbf{p}^\star$ of the NEP are thus the fixed points of the waterfilling mapping (cf. the section "Nash Equilibrium Problems").

The existence of a solution of an NE, for any given set of channels and power budgets of the users, follows readily from results in the section "Existence and Uniqueness of the NE Based on VI." The NEP (29) indeed satisfies the existence conditions given in (18). The study of uniqueness of the NE as well as convergence of algorithms can be addressed using results in the sections "Existence and Uniqueness of the NE Based on VI" and "Algorithms for Nash Equilibria," respectively, based on the nonlinear VI reformulation of the game. We leave the reader the easy task of specializing these results to the NEP (29). Here, we briefly illustrate the alternative formulation of the NEP as a linear VI, mentioned earlier. More specifically, in [6] the authors showed that the NEP (29) is equivalent to the linear VI($\overline{\mathcal{P}}$, $\mathbf{F}$), where $\overline{\mathcal{P}} = \overline{\mathcal{P}}_1 \times \cdots \times \overline{\mathcal{P}}_{\mathcal{Q}}$, with each $\overline{\mathcal{P}}_i$ defined as $\mathcal{P}_i$ in (27) except for the power constraint to be satisfied with equality, and $\mathbf{F}(\mathbf{p}) \triangleq (\mathbf{F}_i(\mathbf{p}))_{i=1}^{Q}$, with

$$\mathbf{F}_i(\mathbf{p}) = \boldsymbol{\sigma}_i + \sum_{j=1}^{Q} \mathbf{M}_{ij} \mathbf{p}_j, \tag{31}$$

where

$$\boldsymbol{\sigma}_i \triangleq \left(\frac{\sigma_i^2(k)}{|H_{ii}(k)|^2}\right)_{k=1}^{N} \text{ and } \mathbf{M}_{ij} \triangleq \mathrm{diag}\left\{\left(\frac{|H_{ij}(k)|^2}{|H_{ii}(k)|^2}\right)_{k=1}^{N}\right\}.$$

This reformulation of the NEP (29) has the following important implications. First, we can readily obtain conditions guaranteeing the uniqueness of the NE invoking result iii) of (13): $\mathbf{F}(\mathbf{p})$ in (31) is strongly monotone on $\overline{\mathcal{P}}$ if and only if $\mathbf{M} \triangleq (\mathbf{M}_{ij})_{i,j=1}^{Q}$ is positive definite [see the section "Existence and Uniqueness of the Solution"]. Rearranging the diagonal blocks $\mathbf{M}_{ij}$ of $\mathbf{M}$, it is not difficult to see that $\mathbf{M}$ is positive definite if so are all the matrices $\mathbf{M}(k) \triangleq (|H_{ij}(k)|^2/H_{ii}(k)|^2)_{i,j=1}^{Q}$. These conditions have an interesting physical interpretation: The uniqueness of the NE is ensured if the interference among the users is sufficiently small (see, e.g., [35], [9], and [8]).

The second important implication of the linear VI reformulation of the NEP is that it provides a geometric interpretation of the waterfilling solution in (30) (also proved independently in [35] and [8]) which is the key point to prove global convergence of all the iterative algorithms based on the waterfilling best response, widely studied in the literature [5], [35], [8], and [6]. More specifically, invoking the equivalence between the VI($\overline{\mathcal{P}}$, $\mathbf{F}$) and the NEP (29) and the characterization of the solution of a VI as given in (15), we have that $\mathbf{p}^\star \in \overline{\mathcal{P}}$ is an NE if and only if

$$\mathbf{p}_i^\star = \mathrm{wf}_i(\mathbf{p}_{-i}^\star) = \prod_{\overline{\mathcal{P}}_i}\left(-\boldsymbol{\sigma}_i - \sum_{j \neq i} \mathbf{M}_{ij} \mathbf{p}_j^\star\right) \tag{32}$$

for all $i = 1, \ldots, Q$, which establishes the equivalence between the waterfilling solution (30) and the Euclidean projection of the negative of the noise plus MUI vector onto the polyhedral set $\overline{\mathcal{P}}$.

The interpretation of the waterfilling solution as a projection simplifies the analysis of the convergence of iterative waterfilling-based algorithms. The state-of-the-art algorithm is the totally asynchronous iterative waterfilling algorithm (IWFA) proposed in [9], where the users can update their power allocation according to the waterfilling solution (30) at arbitrary times and possibly using an outdated version of the MUI. The convergence of this general algorithm is indeed proved via contraction arguments using the projection expression of the waterfilling mapping as in (32) and the nonexpansive property of the projection. We refer to [9] for details. In Figure 5, we show an example of application of the sequential and the simultaneous version of Algorithm 1, which are the well-known sequential IWFA and simultaneous IWFA [5], [6], [8], and [35].

The analysis described so far as well as the asynchronous IWFA can be generalized to the case of MIMO ICs. We refer to [11] and [12] for details.

### *GENERALIZED NASH EQUILIBRIUM PROBLEMS: POWER MINIMIZATION GAME WITH QUALITY OF SERVICE CONSTRAINTS OVER INTERFERENCE CHANNELS*

We consider the reverse problem of the game in (29) under the same system model and assumptions: each player competes against the others by choosing the power allocation over the parallel channels that attains the desired information rate, with the minimum transmit power [10]. This game theoretical formulation is motivated by practical applications, where a prescribed quality of service (QoS) in terms of achievable rate $r_i^\star$ for each user needs to be guaranteed. Stated in mathematical terms, we have the following optimization problem for each player $i$ [10]

$$\begin{aligned} \underset{\mathbf{p}_i}{\text{minimize}} \quad & \sum_{k=1}^{N} p_i(k) \\ \text{subject to} \quad & r_i(\mathbf{p}_i, \mathbf{p}_{-i}) \geq r_i^\star, \end{aligned} \tag{33}$$

where the information rate $r_i(\mathbf{p}_i, \mathbf{p}_{-i})$ is defined in (29).

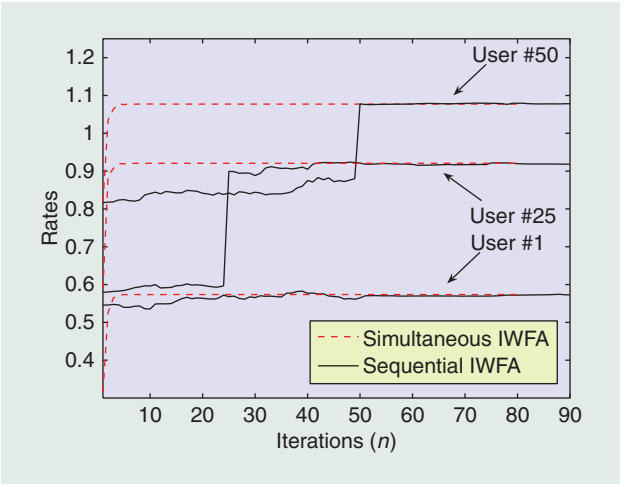
The game in (33) is an example of GNEP in the general form (cf. the section "Generalized Nash Equilibrium Problems"). Note that the single user optimization problem in (33), given the power allocation vectors of the others, admits a unique solution, which is the classical waterfilling solution, where the water level is chosen to satisfy the rate constraint in (33) with equality [10]. In spite of this apparent similarity with the NEP (29), the analysis of the GNEP (33) is extremely hard. This is principally due to the nonlinear coupling among the players' strategies and the unboundedness of the users feasible regions. Nevertheless, in [10] the authors provided a satisfactory answer to the characterization of the GNEP. The analysis in [10] is mainly based on a proper nonlinear transformation that turns the GNEP in the power variables into a standard NEP in some new rate variables, thanks to which one can can borrow from the more developed framework of standard VIs for a fruitful study of the game. Due to the complexity of the analysis, we do not go into details and refer the interested reader to [10]. Here, we only point out that, building on the VI framework, the authors provided sufficient conditions for the existence and uniqueness of a solution of the GNE as well as the convergence of the distributed algorithms based on the single user waterfilling solution of (33), namely the sequential and the simultaneous IWFA. Note that, even though the rate constraints induce a coupling among the feasible strategies of all the users, both algorithms are still totally distributed.

### VI FORMULATION: DESIGN OF CR SYSTEMS UNDER TEMPERATURE-INTERFERENCE CONSTRAINTS

We consider a hierarchical CR network composed of $P$ primary users and $Q$ secondary users, each formed by a single transmitter-receiver pair, coexisting in the same area and sharing the same band. We focus on (block) transmissions over SISO frequency-selective channels; more general results valid for MIMO channels can be found in [15] and [16]. Because of the lack of coordination among the CR users, the set of secondary users can be naturally modeled as a frequency-selective $N$-parallel Gaussian IC, where $N$ is the number of available subcarriers, the maximum information rate on link of the secondary pair $i$ is given by $r_i(\mathbf{p}_i, \mathbf{p}_{-i})$ in (28), and the power allocation vector $\mathbf{p}_i = \{p_i(k)\}_{k=1}^N$ is subject to the power constraint $\mathbf{p}_i \in \mathcal{P}_i$, with $\mathcal{P}_i$ defined in (27) (spectral mask can also be included; see [17]).

Opportunistic communications in CR systems enable secondary users to transmit with overlapping spectrum and/or coverage with primary users, provided that the degradation induced on the primary users' performance is null or tolerable [36]. This can be handled, e.g., introducing some interference constraints that impose a upper bound on the per-carrier and total aggregate interference (the interference temperature limit [36]) that can be tolerated by each primary user. For the sake of simplicity, here, we focus only on per-carrier interference constraints imposed by each primary user $p = 1, \ldots, P$ (both per-carrier and total interference constraints are considered in [17])

$$\sum_{i=1}^{Q} |H_{\mathrm{pi}}^{(P,S)}(k)|^2 p_i(k) \le P_{p,k}, \quad \forall k = 1, \ldots, N, \tag{34}$$




**[FIG5] Rates of the users versus iterations achieved using Algorithm 1: sequential IWFA (solid line curves) and simultaneous IWFA (dashed line curves) for a frequency-selective IC composed by $Q = 50$ users. To make the figure not excessively overcrowded, we report only the curves of three out of 50 links.**

where $H_{\mathrm{pi}}^{(P,S)}(k)$ is the channel transfer function between the transmitter of the $i$th secondary user and the receiver of the $p$th primary user and $P_{p,k}$ is the maximum interference over subcarrier $k$ tolerable by the $p$th primary user, respectively. These limits are chosen by each primary user, according to his QoS requirements.

The aim of each secondary user is to maximize his own rate $r_i(\mathbf{p}_q, \mathbf{p}_{-q})$ under the local power constraints in (27) and the additional global interference constraints in (34). To keep the optimization as decentralized as possible while imposing global interference constraints, we consider the following NEP with pricing (see [17] for the motivations in using this game theoretical model): for all $i = 1, \ldots, Q$,

$$\begin{aligned} \underset{\mathbf{p}_i}{\text{maximize}} \quad & r_i(\mathbf{p}_i, \mathbf{p}_{-i}) - \sum_{p=1}^{P}\sum_{k=1}^{N} \lambda_{p,k} |H_{\mathrm{pi}}^{(P,S)}(k)|^2 p_i(k) \\ \text{subject to} \quad & \mathbf{p}_i \in \mathcal{P}_i \end{aligned} \tag{35}$$

where the prices $\boldsymbol{\lambda}_p = \{\lambda_{p,k}\}_{k=1}^N$ are chosen such that the following complementary conditions are satisfied: $\forall p = 1, \ldots, P$ and $\forall k = 1, \ldots, N$,

$$0 \le \lambda_{p,k} \perp P_{p,k} - \sum_{i=1}^{Q} |H_{\mathrm{pq}}^{(P,S)}(k)|^2 p_i(k) \ge 0. \tag{36}$$

These constraints state that the per-carrier constraints must be satisfied together with nonnegative pricing; in addition, they imply that if one constraint is trivially satisfied with strict inequality then the corresponding price should be zero (no punishment is needed in that case).

### VI REFORMULATION

Building on results in the section "GNEPs with Shared Constraints," we can now readily cast (35)–(36) in a VI and

successfully use the VI tool developed in the first part of the article to study the game. Indeed, (35)–(36) are instances of the NEP with pricing (25)–(26), is as follows by a direct comparison. Then, it follows from the equivalence between the latter problem and the VI problem as illustrated in the section "GNEPs with Shared Constraints," that the NEP (35) is equivalent to the VI$(\hat{\mathcal{P}}, \mathbf{F})$, where

$$\hat{\mathcal{P}} \triangleq \mathcal{P} \cap \left\{ \mathbf{p} \in \mathcal{R}^{NQ} : \sum_{i=1}^{Q} |H_{\mathrm{pi}}^{(P_i,S)}(k)|^2 p_i(k) \le P_{p,k}, \quad \forall\, p\,, k \right\} \tag{37}$$

and $\mathbf{F}(\mathbf{p}) \triangleq (\mathbf{F}_i(\mathbf{p}))_{i=1}^{Q}$, with $\mathbf{F}_i(\mathbf{p}) \triangleq -\nabla_{\mathbf{p}_i} r_i(\mathbf{p}_i, \mathbf{p}_{-i})$. Such a correspondence means that if $\mathbf{p}^\star$ is a solution of the VI$(\hat{\mathcal{P}}, \mathbf{F})$, then there exists a set of prices $\boldsymbol{\lambda}^\star$ such that $(\mathbf{p}^\star, \boldsymbol{\lambda}^\star)$ is an equilibrium pair of the problems (35)–(36); conversely if $(\mathbf{p}^\star, \boldsymbol{\lambda}^\star)$ is a solution of (35)–(36), then $\mathbf{p}^\star$ is a solution of the VI$(\hat{\mathcal{P}}, \mathbf{F})$. According to results in the section "GNEPs with Shared Constraints," the Nash equilibria $\mathbf{p}^\star$ of the NEP (35), the solutions of the VI$(\hat{\mathcal{P}}, \mathbf{F})$, can be interpreted as the variational solutions of the GNEP with shared constraints, having a shared constraint set given in (37) [see (22)] and prices $\boldsymbol{\lambda}^\star$.

**A MORE GENERAL FRAMEWORK SUITABLE FOR INVESTIGATING VARIOUS OPTIMIZATION PROBLEMS AND EQUILIBRIUM PROBLEMS IS THE VARITATIONAL INEQUALITY PROBLEM.**

### EXISTENCE AND UNIQUENESS OF THE NE

Given the equivalence between (35)–(36) and the VI$(\hat{\mathcal{P}}, \mathbf{F})$, it follows readily from the existence results of a solution of the VI [see (8)] that the NEP (35) always admits an NE, for any given set of channels, power budgets of the users, and interference constraints. As far as the uniqueness of the NE is concerned, invoking result iii) in (13) we have that the strong monotonicity of $\mathbf{F}(\mathbf{p})$ on $\hat{\mathcal{P}}$ is a sufficient condition for the uniqueness of the power allocation vector $\mathbf{p}^\star$ at the NE of the NEP. Sufficient conditions guaranteeing the strongly monotonicity of $\mathbf{F}(\mathbf{p})$ are given in [17] and [33]. These conditions have an intuitive physical interpretation: The uniqueness is guaranteed if the interference among the secondary users is not too high.

**ALGORITHM 2: PROJECTION ALGORITHM WITH CONSTANT STEP SIZE**

(S.0): Choose any $\boldsymbol{\lambda}^{(0)} \ge 0$, and the step size $\tau > 0$, and set $n = 0$.

(S.1): If $\boldsymbol{\lambda}^{(n)}$ satisfies a suitable termination criterion: STOP

(S.2): Given $\boldsymbol{\lambda}^{(n)}$ compute $p^\star(\boldsymbol{\lambda}^{(n)})$ as the NE solution of the NEP (35) with fixed prices $\boldsymbol{\lambda} = \boldsymbol{\lambda}^{(n)}$;

(S.3): Update the price vectors: for all $p = 1, \ldots, P$ and $k = 1, \ldots, N$, compute

$$\lambda_{p,k}^{(n+1)} = \left[ \lambda_{p,k}^{(n)} - \tau \left( P_{p,k} - \sum_{i=1}^{Q} |H_{pi}^{(P,S)}(k)|^2 \, [p_i^\star(\boldsymbol{\lambda}^{(n)})]_k \right) \right]^+, \tag{38}$$

(S.4): Set $n \leftarrow n + 1$; go to (S.1).

### DISTRIBUTED ALGORITHMS

Many alternative algorithms have been proposed in [17] to solve the VI$(\hat{\mathcal{P}}, \mathbf{F})$ and thus (35)–(36), that differ in the signaling among primary and secondary users, computational effort, and convergence speed. As an example, here, we consider the projection algorithm (with constant step size) [22, Alg. 12.1.4], formally described in Algorithm 2.

The algorithm can be interpreted as follows. In the main loop, at the $n$th iteration, each primary user $p$ measures the received interference generated by the secondary users and, locally and independently from the other primary users, adjusts his own set of prices $\boldsymbol{\lambda}_p^{(n)}$ accordingly, via a simple projection scheme [see (38)]. The primary users broadcast their own prices $\boldsymbol{\lambda}_p^{(n)}$'s to the secondary users, who then play the game in (35) keeping fixed the prices to the value $\boldsymbol{\lambda}^{(n)}$. The Nash equilibria of such a game can be reached by the secondary users using any algorithm falling in the class of asynchronous IWFA [9] as, e.g., Algorithm 1, whose convergence is guaranteed under mild conditions given in [17] and [9]. Note that, keeping fixed the prices $\boldsymbol{\lambda}$, the (unique) solution of each optimization problem in (35) has a (multilevel) waterfilling-like expression and thus can be efficiently and locally computed. Building on convergence results of the projection algorithm for the VI$(\hat{\mathcal{P}}, \mathbf{F})$ [22], one can prove that, under the same sufficient conditions guaranteeing the strongly monotonicity of $\mathbf{F}$ on $\mathcal{P}$ (see [17]), Algorithm 2 asymptotically converges to a solution of VI$(\hat{\mathcal{P}}, \mathbf{F})$, provided that the step size $\tau > 0$ is chosen arbitrarily but smaller than a prescribed value (given in [17]).

As a numerical example, in Figure 6 we compare three different approaches, namely the NEP formulation with pricing (Algorithm 2), the classical IWFA [5], and the IWFA with individual spectral mask constraints [8], [9], in terms of interference generated at the primary user receivers and the achievable average sum-rate from the secondary users. We refer to these algorithms as flexible IWFA, classical IWFA, and conservative IWFA, respectively. As an example, we consider a CR system composed of six secondary links randomly distributed within an hexagonal cell and one primary user (the base station at the center of the cell). The primary user imposes a constraint on the maximum interference that can tolerate, assumed for simplicity constant over the whole spectrum, i.e., $P_{p,k} = 0.01$ for all $k = 1, \ldots, N$ [see (34)]. The spectral mask constraints used in the conservative IWFA are chosen so that all the secondary users generate the same interference level at the primary receiver and the aggregate interference satisfies the imposed interference threshold. In Figure 6(a), we plot the PSD of the interference generated by the secondary users at the receiver of the primary user, obtained using the aforementioned algorithms. We clearly see from the picture that while classical IWFA violates the interference constraints, both conservative and flexible IWFAs satisfy them, but the global interference constraints impose less

stringent conditions on the transmit power of the secondary users than those imposed by the individual interference constraints based on the spectral masks. However, this comes at the price of some signaling from the primary to the secondary users. Thanks to less stringent constraints on the transmit powers of the secondary users, the flexible IWFA is expected to exhibit a much better performance than the conservative IWFA also in terms of rates achievable by the secondary user. Figure 6(b) confirms this intuition.

### *POTENTIAL GAMES: FLOW AND CONGESTION CONTROL IN MULTIHOP COMMUNICATION NETWORKS*

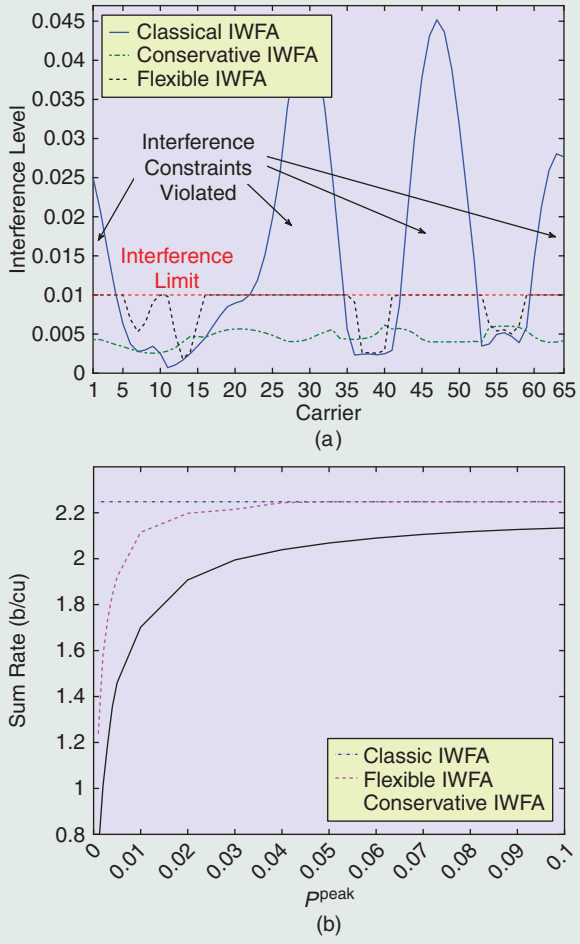
In this section, we complete the picture of the use of game theory and variational inequalities by showing that VI reformulations of a broad class of distributed flow and congestion control, pricing, and resource allocation problems in communication networks with a fairly general topology are possible and allow the recovery of easily known results and establishes new ones along with suitable solution algorithms. An overview of a direct application of game theoretical results to model and solve several instances of the aforementioned problems in telecommunication networks can be found, e.g., in [18] and [19].

We consider a general network model based on fluid approximation. The topology of the network is characterized by a set of nodes $\mathcal{V} = \{1, \ldots, V\}$ and a set of links $\mathcal{L} = \{1, \ldots, L\}$ connecting the nodes (we assume that the network is connected). There are $Q$ active users (players); each user $i$ is uniquely associated to a connection between the source node $s_i$ and the destination node $d_i$ through a path $\mathcal{L}_i$ (predetermined by a routing algorithm), where $\mathcal{L}_i$ is the subset of links that form the path of user $i$. The information flow routed through the path $\mathcal{L}_i$ by user $i$ is denoted by $x_i$ and it holds $0 \le x_i \le x_i^{\max}$, where $x_i^{\max}$ is a physical or regulatory positive upper bound. Each link $\ell$ has a capacity constraint $c_\ell$. If we introduce the $L \times Q$ routing matrix $\mathbf{A}$, defined as $A_{\ell,i} = 1$ if $\ell \in \mathcal{L}_i$ and 0 otherwise, the capacity constraints can be expressed in vector form as $\mathbf{Ax} \le \mathbf{c}$, where $\mathbf{c} = (\mathbf{c}_\ell)_{\ell=1}^{L}$. Finally, we define the set $\mathcal{Q}$ of shared constraints as $\mathcal{Q} \triangleq \{\mathbf{x} \in \mathcal{R}^Q : \mathbf{Ax} \le \mathbf{c}, \ \mathbf{0} \le \mathbf{x} \le \mathbf{x}^{\max}\}$.

We can associate with this setting a GNEP with shared constraints (cf. the section "Generalized Nash Equilibrium Problems"), where $\mathcal{Q}$ is the feasible set and the payoff function of each player $i$ is

$$f_i(\mathbf{x}) = \sum_{\ell:\ell\in\mathcal{L}_i} P_\ell(\mathbf{x}) - U_i(x_i), \tag{39}$$

which is taken as the difference of a pricing function (the sum of the costs relative to each link on the path $\mathcal{L}_i$) and a reward $U_i$ associated to $x_i$, the flow sent by the player. The first term in the payoff function can be interpreted as the price that each user pays for using the network resources. We assume that each $P_\ell$ depends only on the sum of the flows on that link (the traffic on that link): $P_\ell(\mathbf{x}) = P_\ell\big(\sum_{j:\ell\in\mathcal{L}_j} x_j\big)$, with $P_\ell$ a convex function defined on $[0, c_\ell]$. The utility function $U_i$, instead, is assumed to be, according to standard economic conditions and elastic traffic




**[FIG6] Comparison of IWFA algorithms: classical IWFA , conservative IWFA, and flexible IWFA. (a) PSD of the interference profile at the primary user's receiver. (b) Achievable average sum-rate versus the interference constraint.**

model, a strictly concave function defined on $[0, x_i]$. Several pricing and reward functions have been proposed in the literature that satisfy the above assumptions (see, e.g., [19] and references therein). Typical examples are

$$P_\ell = \frac{b_\ell}{\varepsilon + c_\ell - \sum_{j:\ell\in L_j} x_j}, \quad \text{and} \quad U_i = a_i \log(1 + x_i),$$

where $\varepsilon$ is a given positive constant. Note that, under our assumptions, the objective functions $f_i(x_i, x_{-i})$ are strictly convex in $x_i$ for every fixed $x_{-i}$ and convex in $\mathbf{x}$.

#### VI REFORMULATION

We focus on variational equilibria of the GNEP defined above as, in this setup, they are particularly significant. According to results in the section "GNEPs with Shared Constraints," the variational equilibria are the solutions of the VI$(\mathcal{Q}, \mathbf{F})$, where $\mathcal{Q}$

is defined above and $\mathbf{F}(\mathbf{x}) \triangleq (\nabla_{x_i} f_i(\mathbf{x}))_{i=1}^{Q}$. Since the feasible set $\mathcal{Q}$ is convex and compact, we have that the VI $(\mathcal{Q}, F)$ has at least one solution [see (8)]. This equilibrium is therefore also a solution of the GNEP. The interesting point is that in this case the function $\mathbf{F}$ is the gradient of a scalar function $f(\mathbf{x})$, i.e., $\mathbf{F}(\mathbf{x}) = \nabla f(\mathbf{x})$, with

$$f(\mathbf{x}) = \sum_{\ell \in \mathcal{L}} P_\ell\left(\sum_{j:\ell \in \mathcal{L}_j} x_j\right) - \sum_{i=1}^{Q} U_i(x_i). \tag{40}$$

Hence, the variational solutions are the solutions of the VI $(\mathcal{Q}, \nabla f)$ and therefore, by the results in the section "Variational Inequalities Problems," the solutions of the optimization problem

$$\begin{aligned} &\underset{\mathbf{x}}{\text{minimize}} \quad f(\mathbf{x}) \\ &\text{subject to} \quad \mathbf{x} \in \mathcal{Q}. \end{aligned} \tag{41}$$

Equation (41) can be viewed as a centralized "system problem" where one tries to minimize the costs over all the links minus the total rewards for the players. Since $f$ is strictly convex, (41) has a unique solution and so does the equivalent VI $(\mathcal{Q}, \nabla f)$ (cf. the section "Variational Inequalities Problems"). Therefore, we have shown that the original GNEP has at least a solution, even if it can actually have more then one solution, but the same game has one and only one variational solution that turns out to be also the unique solution of (41). This variational solution, being at the same time a solution of the game and a "system" optimum, is a desirable outcome of the GNEP.

## DISTRIBUTED ALGORITHMS

In principle, one can use standard decomposition methods for the solution of (41) and get convergence to the variational equilibrium, However, such standard methods, for example, the alternating method of multipliers, require a considerable coordination and exchange of information among the players, which makes them not appealing in noncooperative scenarios. We already mentioned in the section "Generalized Nash Equilibrium Problems" that the development of decentralized algorithms for GNEPs with shared constraints is not a trivial task. In our case, due to the fact that the game is a "potential game" (i.e., $\mathbf{F} = \nabla f$), we can apply the regularized Gauss-Seidel method proposed in [37]. This is a modification of the Gauss-Seidel method considered in Algorithm 1 in the section "Algorithms for Nash Equilibria" and is formally described in Algorithm 3: the players solve in sequence their own minimization problem, taking the other players' variables as given and adding a regularization term to their objective function.

The results in [37] show that this decomposition procedure generates a sequence contained in $\mathcal{Q}$ such that every limit point is a NE. The only drawback is that we cannot guarantee convergence to the variational equilibrium.

**ALGORITHM 3: REGULARIZED GAUSS-SEIDEL ALGORITHM**

(S.0): Choose any feasible starting point $\mathbf{x}^{(0)} = (x_i^{(0)})_{i=1}^{Q}$, the step-size $\tau > 0$, and set $n = 0$.

(S.1): If $\mathbf{x}^{(n)}$ satisfies a suitable termination criterion: STOP

(S.2): for $i = 1, \ldots, Q$, compute a solution $x_i^{(n+1)}$ of

$$\begin{aligned} &\underset{x_i}{\text{minimize}} \quad f_i(x_1^{(n+1)}, \ldots, x_{i-1}^{(n+1)}, x_i, x_{i+1}^{(n)}, \ldots, x_Q^{(n)}) + \tau \|x_i - x_i^{(n)}\|^2 \\ &\text{subject to } \mathbf{A}(x_1^{(n+1)}, \ldots, x_{i-1}^{(n+1)}, x_i, x_{i+1}^{(n)}, \ldots, x_Q^{(n)})^T \le \mathbf{c} \\ &\qquad 0 \le x_i \le x_i^{\max} \end{aligned} \tag{42}$$

end

(S.3): Set $\mathbf{x}^{(n+1)} \triangleq (x_i^{(n+1)})_{i=1}^{Q}$ and $n \leftarrow n+1$; go to (S.1).

## CONCLUSIONS

In this article, we have provided a unified view of some basic theoretical foundations and main techniques in convex optimization, game theory, and VI theory. We put special emphasis on the generality of the VI framework, showing how it allows to tackle several interesting problems in nonlinear analysis, classical optimization, and equilibrium programming. In particular, we showed the relevance of the VI theory in studying Nash and GNE problems. The first part of the article was devoted to provide the (basic) theoretical tools and methods to analyze some fundamental issues of an equilibrium problem, such as the existence and uniqueness of a solution and the design of iterative distributed algorithms along with their convergence properties. The second part of the article made these theoretical results practical by showing how the VI framework can be successfully applied to solve several challenging equilibrium problems in ad hoc wireless (peer-to-peer wired) networks, in the emerging field of CR networks, and in multihop communication networks.

We hope that this introductory article would serve as a good starting point for readers to apply VI theory and methods in their applications, as well as to locate specific references either in applications or theory.

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